(3 Hours) [Total Marks : 100]

N.B. **1.** All Questions are compulsory.
2. Figures to the right indicate marks to respective parts.

Q.1 Choose correct alternative in each of the following: (20)

i. Let $\mathbf{A}=\begin{pmatrix}1 & 2 & -1\\ 3 & -1 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3 & -1\\ 4 & 5\\ 1 & 2\end{pmatrix}$ then the standard matrix for the transformation T defined by $T(\mathbf{x})=\mathbf{A}(\mathbf{B}\mathbf{x})$ is

(a) $\begin{pmatrix}0 & 7 & 4\\ 19 & 3 & 1\\ 7 & 0 & 1\end{pmatrix}$ (b) $\begin{pmatrix}0 & 7 & -3\\ 19 & 3 & 1\\ 7 & 0 & 1\end{pmatrix}$

(c) $\begin{pmatrix}10 & 7\\ 6 & 5\end{pmatrix}$ (d) $\begin{pmatrix}10 & 7\\ 6 & -6\end{pmatrix}$

ii. The range of the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by the matrix $\mathbf{A}=\begin{pmatrix}2 & 3\\ -6 & -9\end{pmatrix}$ is

(a) All of $\mathbb{R}^2$ (b) A line through the origin having slope -3

(c) A line through the points (2,3) and (-6,-9) (d) A line through origin having slope 2/3

iii. Let T be the linear transformation that T(1, 0) = (4, 3) and T(0, 1) = (7, 2). Then T(3,-2) =

(a) (12,-6) (b) (-14,-4)

(c) (11,5) (d) (-2,5)

iv. Area of the parallelogram spanned by vectors $(-1,2)$ and $(3,4)$ is

(a) 10 (b) 5

(c) 2 (d) None of the above

v. If $E=\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ then E^{-1} is

(a) $\begin{pmatrix}1 & 0 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ (b) $\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

(c) $\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ (d) None of the above

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vi. Which of the following is false for an invertible $n \times n$ matrix A?

(a) $\det A = \det A^t$ (b) $\det A = \det A^{-1}$

(c) $\det A^2 = (\det A)^2$ (d) None of the above

vii. If I_{23} , $I_{13} \in M_3(\mathbb{R})$ then $I_{23} + I_{13}$ is

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

viii. Which of the following set forms group under the given binary operation

(a) $(\mathbb{N}, +)$ (b) $(\mathbb{Q}, +)$

(c) $(\mathbb{Z}, \cdot)$ (d) $(\mathbb{Z}^*, \cdot)$

ix. The order of the group S_n is

(a) $\frac{n!}{2}$ (b) $2n$

(c) $n!$ (d) None of these

x. The identity element of the group $G = \{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} / a \in \mathbb{R}^* \}$ under multiplication of 2×2 matrices is

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Let V,W be vector spaces over $\mathbb{R}$ and T:V→W be a linear transformation and if V is finite dimensional then show that dim V = dim Ker T + dim Img T.

ii. Show that the following are equivalent for a linear map T: V→V ,

1. T is bijective
2. Ker T = {0}
3. Img T = V

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b) Attempt any TWO questions from the following: (12)

i. Let $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined as $T(\mathbf{x}) = \mathbf{Ax}$ where $\mathbf{A}$ is the matrix of T with respect to standard bases $\{e_1, e_2\}$ on both sides and $\mathbf{A}=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. What is the matrix of T with respect to changed bases $\{e_1+e_2, e_2\}$ on both sides?

ii. Let $T:\mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined as $T(\mathbf{x}) = \mathbf{Ax}$ where $\mathbf{A}=\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$. Determine rank T, nullity of T and hence verify the rank – nullity theorem.

iii. Let V be the vector space of real polynomials in the variable x and let $D^3 : V \rightarrow V$ defined as $D^3(f) = \frac{d^3}{dx^3} f$ then find ker D^3. Also find its dimension.

iv. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ the matrix of linear transformation $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as $T(\mathbf{x}) = \mathbf{Ax}$ then show that T is invertible and find formula for T^{-1}.

Q.3 a) Attempt any ONE question from the following: (08)

i. Let $\phi: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bilinear function such that $\phi(A^1, A^1) = 0, \forall A^1 \in \mathbb{R}^2$ and $\phi(E^1, E^2) = 1$ where E^1, E^2 are the standard unit vectors of $\mathbb{R}^2$. Prove that $\phi(A^1, A^2) = \det(A^1, A^2)$ for any column vectors $A^1, A^2 \in \mathbb{R}^2$.

ii. Prove that the row rank and the column rank of an $m \times n$ matrix A are equal.

b) Attempt any TWO questions from the following: (12)

i. Express $\mathbf{A}$ as product of elementary matrices where $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$.

ii. Define row rank ,column rank of $\boldsymbol{A} \in M_{m\times n}(\mathbb{R})$.
Find rank of $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 & 1 \\ 3 & 4 & -1 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}$.

iii. If $A^1, A^2, \cdots, A^n$ are n linearly dependent column vectors in $\mathbb{R}^n$, then prove that $\det(A^1, A^2, \cdots, A^n) = 0$.

iv. Solve the following system of linear equations using Cramer's rule

$$x + y + 2z = 1, \qquad 2x + 4z = 2, \qquad 3y + z = 3$$

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Q.4 a) Attempt any ONE question from the following: (08)

i. Define Group. Prove that the set of residue classes modulo $n, \mathbb{Z}_n$ is a group under addition modulo n.

ii. Let G be a group. For $a, b, x \in G$ Prove that

i) $o(a) = o(xax^{-1})$

ii) $o(ab) = o(ba)$

b) Attempt any TWO questions from the following: (12)

i. Let G be a group and H be non-empty subset of G. Prove that H is a subgroup of G iff $ab^{-1} \in H, \forall a, b \in H$.

ii. Consider $D_3 = \{e, a, a^2, b, ba, ba^2\}$ where $a^3 = e, b^2 = e$ and $ab = ba^2$. Prove that D_3 is a group under composition of functions. Is it abelian?

iii. Construct composition tables of $U(8)$ and $U(10)$ groups. Are they examples of Klein-4 group? Justify your answer.

iv. Let G be a group. For any $a, b \in G$, prove that

1) $(a^{-1})^{-1} = a$

2) $(ab)^{-1} = b^{-1}a^{-1}$

Q.5 Attempt any FOUR questions from the following: (20)

a) Show that linear map $T:\mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x, y) = (2x + y, x + y, x)$ represents one-one linear transformation and linear map $S:\mathbb{R}^3 \to \mathbb{R}^2$ given by $S(x, y, z) = (x + z, x + y)$ represents onto linear transformation.

b) Show that $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x,y) = (x+2y, x-y)$ is a linear isomorphism.

c) Determine the value of k for which the following system of linear equations has no solution:

$x + y + z = 2, \quad x + 2y + z = -2, \quad x + y + (k-5)z = k$

d) Using adjoint of a matrix, find inverse of $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 5 & 4 \\ 3 & 6 & 9 \end{pmatrix}$.

e) Construct composition table of $\mathbb{Z}_5^*$ under multiplication modulo 5. Also find order of all elements of $\mathbb{Z}_5^*$.

f) Show that $H = \{ I_4, (12)\circ(34), (13)\circ(24), (14)\circ(23)\}$ is a subgroup of S_4