(3 Hours) **[Total Marks: 100**

N.B.: 1. All questions are compulsory.

2. Figures to the right indicate full marks.

Q.1 Choose the correct alternative in each of the following: (20)

i. For which value of k does the following system have infinitely many solutions?
$$\begin{aligned} 2x - y &= k \\ 4x - 2y &= 6 \end{aligned}$$

(a) 3 (b) 6

(c) 1 (d) 2

ii. Which of the following matrix is skew symmetric?

(a) $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 7 & 0 \\ 7 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii. The parametric representation for the line through the points (1,2) and (3,2) is

(a) $x = 4 + 4t, y = -1 + t$ (b) $x = 2 - t, y = 2 - 3t$

(c) $x = 1 + 2t, y = 2 + t$ (d) None of these

iv. Which of the following sets is linear independent?

(a) {(1,-1), (1,0)} (b) $\{(\frac{3}{2}, 3), (3,6)\}$

(c) {(-4,2), (-8,4)} (d) {(1,3), (-2,-6)}

v. Which of the following is not a subspace of $\mathbb{R}^2$ over $\mathbb{R}$?

(a) $\{(x, y) \in \mathbb{R}^2 | \, x = 0\}$ (b) $\{(x, y) \in \mathbb{R}^2 | \, x + y = 0\}$

(c) $\{(x, y) \in \mathbb{R}^2 | x + 2y = 0\}$ (d) $\{(x, y) \in \mathbb{R}^2 | x - y = 3\}$

vi. Which of the following is a generating set of $\mathbb{R}^2$?

(a) $\{(2,3), (0,6)\}$ (b) $\{(1,3), (2,6)\}$

(c) $\{(3,0), (0,0)\}$ (d) $\{(1, -1), (3, -3)\}$

vii. The rank of linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x, y) = (x - y, x - y)$ is

(a) 2 (b) 3

(c) 4 (d) 1

viii. The nullity of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined as $T(x, y, z) = (0,\ 0)$ is

(a) 1 (b) 0

(c) 2 (d) 3

ix. If for a linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^4$ the Dim(Ker(T)) = 1 then the Rank of T is

(a) 2 (b) 1

(c) 3 (d) 4

x. Which of the following is the basis for $W = \{(x, y, z) \in \mathbb{R}^3 |\ x + y + 2z = 0\}$?

(a) {(-1,1,0), (-2,0,1)} (b) {(0,1,-1), (0,1,0)}

(c) {(1, 0, -2), (1,1,1)} (d) {(1, -1, 1), (2, 1, -1)}

Q.2 a) Attempt any ONE question from the following: (08)

i. If A and B are $n \times n$ matrices then prove that

(1) $(A + B)^T = A^T + B^T$.

(2) If A is invertible prove that A^T is invertible and $(A^{-1})^T = (A^T)^{-1}$

ii. For $m, n \in \mathbb{N}$ using induction on m prove that any homogeneous system of m real linear equations in n unknowns has a non-trivial solution if $m < n$.

b) Attempt any TWO questions from the following: (12)

i. Let A=(a_{ij}) be $n \times n$ real matrix and $X = (x_1, x_2, \ldots\ldots, x_n)^t$.

If $\alpha = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)^t$ and $\beta = (\beta_1, \beta_2, \ldots\ldots, \beta_n)^t$,

are solutions of the linear homogeneous system $AX = 0$,

then prove that $\alpha + \beta$ and $k\alpha,\ \alpha \in \mathbb{R}$ are also solutions of the same system.

ii. Use parametric equations of line to check if the points $(1, -2, 0), (2, 0, -3)$and (4, 4, - 6) are collinear.

iii. Geometrically interpret solutions of the real linear homogenous system of 2 equations in 3 unknowns.

iv. Solve the system of linear equations: $x + y + z = 3$, $x + 2y + 2z = 5,\ 3x + 4y + 4z = 12$ using Gaussian Elimination Method.

Q.3 a) Attempt any ONE question from the following: (08)

i. Prove the following properties of a real vector space V:

1. $a.0_v = 0_v, \forall a \in \mathbb{R}$ and $\forall v \in V$.
2. $0.v = 0_v, \forall v \in V$.
3. $a.v = 0_v$ implies either $a = 0$ or $v = 0_v$, $\forall a \in \mathbb{R}$ and $\forall v \in V$.

ii. Let V be a real vector space and $S = \{u_1, u_2, \ldots, u_n\} \subseteq V$ .Show that S is linear dependent if and only if one of the vectors in S can be written as a linear combination of the other vectors in S.

b) Attempt any TWO questions from the following: (12)

i. Let $V = M_2(\mathbb{R})$ and $W = \left\{ A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in V \,\middle|\, a, b, c \in \mathbb{R} \right\}$. Show that W is vector subspace of V.

ii. Check whether $\{(1,3), (4,0), (9,15)\}$ is a linear independent set in $\mathbb{R}^2$.

iii. Prove the following properties of a real vector space:
1) V has a unique additive identity.
2) Every vector in V has a unique additive inverse.

iv. Define a linear span of a non-empty subset of a real vector space. Let $V = P_2[x] = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$. Express vector $1 + 2x + x^2$ in V as a linear combination of $1 + x, x^2, x + x^2$ in V.

Q.4 a) Attempt any ONE question from the following: (08)

i. Show that every finitely generated vector space has a basis.

ii. Let V and W be real vector spaces over and $T: V \to W$ be a linear transformation. Prove that $\operatorname{Ker} T$ is a subspace of V and $Image\ T$ is a subspace of W.

b) Attempt any TWO questions from the following: (12)

i. Check if the set $\{(1,0,1),(1,1,0),(1,0,-1)\}$ is a basis of $\mathbb{R}^3$.

ii. Find the dimension of image space of $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x,x,y)$.

iii. Find the matrix associated with the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T(x,y,z) = (x+y,x,y,z)$ with respect to standard bases of $\mathbb{R}^3$ *and* $\mathbb{R}^4$.

iv. Find the basis of the subspace $W = \{(x,y) \in \mathbb{R}^2 \mid x+y=0\}$ and extend it to a basis of $\mathbb{R}^2$.

Q.5 Attempt any FOUR questions from the following: (20)

a) Find the value of k so that the system of linear equations $x + y + 2z = 1,\ x + 2y - z = 2,\ 2x + 3y + z = k$ becomes consistent.

b) Transform the following matrix to it's row echelon form :

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 1 & 3 & -2 & 3 & 0 \\ 2 & 4 & -3 & 6 & 4 \\ 1 & 1 & -1 & 4 & 6 \end{bmatrix}$$

c) Let V be vector space of all real valued sequences and $S = \{(x_n) \in V : (x_n) \text{ is convergent}\}$. Show that S is a subspace of V over $\mathbb{R}$.

d) Examine whether the set S generates $\mathbb{R}^3$ where S = { (2,3,0) , (1,0,4) , (0,2,0) }

e) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $T(1,2) = (3,-3)$ *and* $T(3,2) = (-2,1)$. Find $T(5,6)$.

f) Verify the Rank-Nullity theorem for $T : \mathbb{R}^2 \to \mathbb{R}$ defined as $T(x,y) = x + y$.

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