Q.P.Code: 51639

(3 Hours) **[Total Marks : 100]**

N.B.: 1. All questions are compulsory.
2. Figures to the right indicate full marks.

Q.1 Choose correct alternative in each of the following: (20)

i. The series $\sum_{n=1}^{\infty} \frac{5n+2}{4^n}$ of real numbers

(a) is not convergent (b) Converges to $\frac{5}{4}$

(c) Converges to 0 (d) None of the above

ii. If $\sum x_n$ and $\sum y_n$ are two series of real numbers such that $\sum(x_n - y_n)$ and $\sum y_n$ are both convergent then $\sum x_n$

(a) Is convergent (b) Is divergent

(c) Conditionally convergent (d) None of the above

iii. The series $\sum_{n=1}^{\infty}(6r)^n, r \in \mathbb{R}$ is

(a) Convergent for any $r \in \mathbb{R}$ (b) Divergent for any $r \in \mathbb{R}$

(c) Convergent if $|r| < \frac{1}{6}$ (d) None of the above

iv. The function $y = \frac{1}{x+1}$ has $\frac{d^4y}{dx^4}$ equal to

(a) $\frac{1}{(x+1)^5}$ (b) $\frac{64}{x+1}$

(c) $\log(x+1)$ (d) None of the above

v. The function $f(x) = |x+5|, x \in \mathbb{R}$

(a) Is differentiable at $x = -5$ (b) Is not differentiable at $x = 5$

(c) Is differentiable at every $x \in \mathbb{R}$ (d) None of the above

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vi. The value of $\lim_{x\to 0}\frac{(8^x-3^x)}{x}$ is

(a) $\log_e\left(\frac{8}{3}\right)$ (b) $\log_{10} 5$

(c) $\frac{8}{3}$ (d) None of the above

vii. The function $f(x)=\frac{1}{x-5}$, $\forall\, x\in(5,6)$ is

(a) Continuous and bounded (b) Continuous but not bounded

(c) Discontinuous (d) None of the above

viii. The function $f(x)=\log x\,, x>0$ is

(a) Concave upwards (b) Concave downwards

(c) Decreasing function (d) None of the above

ix. The function $f(x)=3x^2-7x+2$ is

(a) Increasing for all $x\in\mathbb{R}$ (b) Decreasing for all $x\in\mathbb{R}$

(c) Increasing for all $x>\frac{7}{6}$ (d) None of the above

x. If $f,g:\mathbb{R}\to\mathbb{R}$ are such that $f\cdot g$ is differentiable then

(a) Both f,g are differentiable (b) At least one of f,g is differentiable

(c) $f+g$ is differentiable (d) None of the above

Q.2 a) Attempt any ONE question from the following: (08)

i. Let (s_n) is the sequence of partial sums for series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$. Prove that (s_{2n}) is decreasing sequence and (s_{2n+1}) is increasing sequence. Further prove that

$$\lim_{n\to\infty} s_{2n}=\lim_{n\to\infty} s_{2n+1}=\lim_{n\to\infty} s_n\,.$$

ii. If $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = 0$ and $\sum_{n=1}^{\infty} y_n$ is absolutely convergent series then prove that series $\sum_{n=1}^{\infty} x_n$ is also convergent.

b) Attempt any TWO questions from the following: (12)

i. Prove that $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k+3}$ is conditionally convergent series.

ii. State the ratio test and use it to test the convergence of $\sum_{k=1}^{\infty}\frac{2^n}{n!}$.

iii. Prove that for each non-negative integer n,

$$\frac{1}{(2^n+1)^2}+\frac{1}{(2^n+2)^2}+\cdots+\frac{1}{(2^n+2^n)^2}\le\frac{1}{2^n}.$$

And deduce that the sequence of partial sums for series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is bounded.

iv. Let $A,B\in\mathbb{R}$, $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Then prove that $\sum_{n=1}^{\infty}(Ba_n - Ab_n) = 0$.

Q.3 a) Attempt any ONE question from the following: (08)

i. Let f be a real valued continuous function on $[a, b]$ such that $f(a) \neq f(b)$. Then for each k, $f(a) < k < f(b)$, prove that there exists $c \in (a, b)$ such that $f(c) = k$.

ii. State and prove Chain Rule for the derivatives of a composite function.

b) Attempt any TWO questions from the following: (12)

i. Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3xy$.

ii. For $y = e^x \sin x$, prove that $y_2 - 2y_1 + 2y = 0$. Hence prove that $y_{n+2} - 2y_{n+1} + 2y_n = 0$.

iii. Prove that $x^3 - 15x + 1 = 0$ has at least one root in $[-4, 4]$.

iv. Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos x$. Show that f is differentiable on $\mathbb{R}$.

Q.4 a) Attempt any ONE question from the following: (08)

i. State and prove Rolle's theorem.

ii. If f is a differentiable function defined on an open interval (a, b) and $f'(x) < 0 \; \forall \, x \in (a, b)$, then prove that f is decreasing on (a, b).

b) Attempt any TWO questions from the following: (12)

i. Find the local maximum and minimum of the function $f(x) = x + \frac{1}{x}$ if they exist.

ii. Use Rolle's theorem to show that the equation $x^3 + x - 1 = 0$ has exactly one real root.

iii. For what values of x is the curve $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ concave upwards and when is it concave downwards? Also find a point of inflection.

iv. Verify Cauchy's Mean Value Theorem for $f(x) = x^2$ and $g(x) = x^3, x \in [1,2]$.

Q.5 Attempt any FOUR questions from the following: (20)

a) Test for convergence of the series $\sum \frac{n^2}{2^n}$ stating the result used.

b) Prove that $\sum x^n$ is convergent if and only if $|x| < 1$.

c) Find n^{th} derivative of $y = x^2 \cos x$.

d) If $f : \mathbb{R} \to \mathbb{R}$ is an even function and differentiable on $\mathbb{R}$ then prove that f' is an odd function.

e) Expand $3x^3 - 2x^2 + 4x + 1$ in powers of $(x - 1)$ using Taylor's theorem.

f) Evaluate $\lim_{x \to 0^+} \frac{\log \tan x}{\log x}$.

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