(3 Hours) [Total Marks : 100]

N.B. **1**. All questions are compulsory.
2. Figures to the right indicate marks for respective parts
3. Use of Calculator is not allowed.

Q.1 Choose correct alternative in each of the following: (20)

i. If $\lim_{n\to\infty} a_n = 0$ then series $\sum_1^\infty a_n$ is

(a) always convergent (b) always divergent
(c) alternating (d) None of the above

ii. If $y = e^{2x+3}$ then y_4 =

(a) $8e^{2x+3}$ (b) $16e^{2x+3}$
(c) 32 (d) none of these

iii. Rolle's theorem is applicable to f(x) = sin x in the interval

(a) $[0, \pi]$ (b) $[0,\frac{\pi}{2}]$
(c) $[\frac{\pi}{2}, \pi]$ (d) none of these

iv. The series $\sum_{n=1}^\infty \frac{(-1)^n}{n}$ of real numbers is

(a) divergent series (b) conditionally convergent series
(c) geometric series (d) none of these

v. The function f(x) = | x – 4 |, x ∈ ℝ is

(a) differentiable at x = 4 (b) not differentiable at x = 4
(c) differentiable at any x in ℝ (d) none of these

vi. Which of the following functions is increasing in [-1, 1] ?

(a) $f(x) = x^2$ (b) f(x) = cos x
(c) f(x) = sin x (d) f(x) = |x|

vii. The series $\sum_{n=1}^\infty 5$ of real numbers is a

(a) divergent series (b) convergent series
(c) alternating series (d) none of these

viii. Amongst the following, the function which has a local minimum at the origin is

(a) y = sin x (b) $y = x^3$
(c) y = |x| (d) $y = x^2 – 2x + 1$

ix. $\lim_{x\to 1} (x-1)^{(x-1)}$ =

(a) 1 (b) 0
(c) e (d) limit cannot be determined

x. The function $f(x) = x^3 + 5x + 1$, x ∈ ℝ is

(a) increasing on ℝ (b) increasing when x > 0
(c) decreasing on ℝ (d) none of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent then the sequence (a_n) converges to zero. Is converse true? Justify your answer.

ii. Prove that the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent if it satisfies the following conditions:

(I) $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ i.e. sequence (a_n) is non-increasing.

(II) $\lim\limits_{n \to \infty} a_n = 0$

b) Attempt any TWO questions from the following: (12)

i. Prove that the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

ii. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of non-negative real numbers. Assume that there exists $n_1 \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq n_1$. Then prove that, if $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

iii. Is the series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ convergent? If yes, find it's limit.

iv. Check whether the following series are convergent stating the results used.

I. $\sum_{n=1}^{\infty}\frac{n+1}{5^n}$, II. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n\sqrt{n}}$.

Q.3 a) Attempt any ONE question from the following: (08)

i. Let $f, g: \mathbb{R} \to \mathbb{R}$ be two functions which are differentiable at $p \in \mathbb{R}$. Prove that fg is differentiable at $p \in \mathbb{R}$.

ii. Let $n \in \mathbb{N}$ and $u, v: \mathbb{R} \to \mathbb{R}$ be n – times differentiable functions. Prove that $(uv)_n = u_n v_0 + \binom{n}{1}u_{n-1}v_1 + \cdots..+\binom{n}{n}u_0 v_n$ where the suffixes denote the order of derivatives and $u_0 = u$ and $v_0 = v$.

b) Attempt any TWO questions from the following: (12)

i. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $H(x) = \begin{cases} \frac{f(x)\sin^2 x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Find $H'(0)$.

ii. If $f: [a, b] \to \mathbb{R}$ is a continuous function then prove that f attains its bounds.

iii. Find the derivative of the following functions using chain rule:

I. $\sqrt{4^x + 1}$ II. $\sin^3 x$

iv. Find $\frac{dy}{dx}$ for $\sin y + x^2y^3 - \cos x = 2y$ where y is a function of x.

Q.4 a) Attempt any ONE question from the following: (08)

i. For a real valued function f define local minimum at a point.
If $f : (a, b) \to \mathbb{R}$ has a local minimum at a point $p \in (a, b)$ and if f is differentiable at p then prove that $f'(p) = 0$

ii. State and prove Rolle's theorem.

b) Attempt any TWO questions from the following: (12)

i. Find the intervals on which $f(x) = 4x^3 - 12x^2 - 36x + 1$ is increasing or decreasing.

ii. State L'Hospital's rule and evaluate $\lim_{x\to 0} \frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$.

iii. Expand $x^3 + 2x + 1$ in powers of $x - 2$.

iv. Determine the intervals of concavity and the inflection points of function $f(x) = 5x^2 - 10x$.

Q.5 Attempt any FOUR questions from the following: (20)

a) Check the following series for absolute and conditional convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{(n^4+1)}$.

b) Check the convergence of the series $\sum_{n=1}^{\infty} \frac{n^3\, 7^n}{n!}$.

c) Check if the following function is differentiable at x = 0

$$f(x) = \begin{cases} x^3 + 1 , & x \le 0 \\ e^x , & x > 0 \end{cases}$$

d) If $y = e^{mx} + e^{-mx}$ then prove that $y_{n+2} = m^2 y_n$.

e) Evaluate $\lim_{x\to 0} x\log(\tan x)$.

f) Find maximum value of $\frac{\log x}{x}$ in $(0, \infty)$.

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