(Time: 3 Hours) **[Total Marks : 100]**

N.B. **1**. All questions are compulsory.

2. Figures to the right indicate marks for respective parts

3. Use of Calculator is not allowed.

Q.1 Choose correct alternative in each of the following: (20)

i. Additive inverse of a real number

(a) Exists and is unique (b) Does not exist

(c) If exists then is unique (d) None of these

ii. If A = (- 4, 7] then

(a) Inf A ∈ A (b) Sup A ∈ A

(c) Inf A ∈ A , sup A∈ A (d) None of these

iii. If 0 < x < 1 then

(a) $x^3 < 1$ (b) $x^3 > 1$

(c) $x^3 > x$ (d) None of these

iv. The sequence (x_n) where $x_n = \sin(n\pi)$, $\forall\, n \in \mathbb{N}$ is

(a) constant (b) decreasing

(c) divergent (d) none of these

v. Every Cauchy sequence in $\mathbb{R}$ is

(a) an increasing sequence (b) divergent

(c) convergent (d) None of these

vi. $\lim_{x \to -2} \frac{x^2-5x-14}{x+2}$ equals

(a) -13 (b) 9

(c) -9 (d) none of these

vii. $\lim_{x \to \infty} \frac{1}{x^2+2x}$ equals

(a) 2 (b) 1

(c) 0 (d) none of these

viii. If sequence (x_n) of real numbers satisfies , $\frac{1}{2n} \leq x_n \leq \frac{1}{n}$, $\forall\, n \in \mathbb{N}$ then (x_n)

(a) converges to 0 (b) diverges

(c) converges to 1 (d) none of these

ix. The graph of a function y= e^x intersects x axis

(a) at (0,0) (b) nowhere

(c) at every point (d) none of these

x. The function f(x) = $2x+3$ is continuous

(a) Only if x > 0 (b) only if x < 0

(c) For each x∈ ℝ (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. State the arithmetic mean and geometric mean (AM-GM) inequality for real numbers. Apply it to prove that $(a+b)(b+c)(c+a) \geq 8abc$ $\forall$ non-negative $a, b, c \in \mathbb{R}$

ii. State and prove the Archimedean order property for ℝ. Hence prove that if real number x satisfies $0 \leq x < \epsilon$ for every positive real ϵ then $x = 0$.

b) Attempt any TWO questions from the following: (12)

i. Prove that $|xy| = |x||y|$ and $||x| - |y|| \leq |x - y|$ $\forall x, y \in \mathbb{R}$.

ii. State the law of trichotomy of real numbers. Hence prove that the square of any non-zero real number is positive.

iii. Find an upper bound, a lower bound , the supremum and the infimum for $\{x : x \in \mathbb{R}, |x-3| \leq 4\}$ if they exist.

iv. State only the Cauchy-Schwartz inequality for ℝ. Apply it to prove that $3(x^2+y^2+z^2) \geq (x+y+z)^2$ $\forall x, y, z \in \mathbb{R}$.

Q.3 a) Attempt any ONE question from the following: (08)

i. Let (x_n) *and* (y_n) be two real sequences such that $(x_n) \to p$ and $(y_n) \to q$, then prove that $(x_n - y_n) \to p - q$

ii. Prove that the real sequence $\left(1+\frac{1}{n}\right)^n$ is convergent in ℝ.

b) Attempt any TWO questions from the following: (12)

i. Prove that every convergent sequence in ℝ is Cauchy.

ii. Use $\epsilon - n_0$ definition to prove that the sequence $\left(\frac{n-6}{7n+4}\right) \to \frac{1}{7}$ $as\ n \to \infty$

iii. Give an example of the following

a) A bounded sequence which is not convergent.

b) Sequences (x_n) and (y_n) such that $(x_n) \to 0$ but $(x_n y_n)$ does not converge to 0.

iv. Prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Q.4 a) Attempt any ONE question from the following: (08)

i. Let f: ℝ → ℝ be a function and let $l \in$ ℝ . When do we say that $\lim_{x\to a} f(x) = l$? Prove that if $\lim_{x\to a} f(x)$ exists then it is unique.

ii. Let f : ℝ → ℝ be a function and p ∈ ℝ. If $(f(x_n))$ converges to f(p) for any sequence (x_n) that converges to p then prove that f is continuous at p.

b) Attempt any TWO questions from the following: (12)

i. Draw the graph of the function $f(x) = x^2 + 1$, for $-2 \le x \le 2$

ii. Using $\epsilon - \delta$ definition, show that the function f (x) = 3x – 2, x ∈ ℝ is continuous at p = 1 .

iii. State Sandwich theorem for limit of functions. Use it to prove that $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$

iv. If $\lim_{x\to a} f(x) = l$ then prove that $\lim_{x\to a} |f(x)| = |l|$

Q.5 Attempt any FOUR questions from the following: (20)

a) Prove that the additive identity in ℝ is unique.

b) If A and B are bounded subsets of ℝ, then prove that $A \cup B$ is bounded in ℝ.

c) Show that sequence $\left(\frac{3}{n}\right)$ is Cauchy in ℝ.

d) Show that the sequence $\left(\frac{n-1}{n+1}\right)$ is monotonic and bounded.

e) Find $\lim_{x\to\infty} \frac{(x-3)(2x^2-5)}{(7x+4)(x^2+1)}$.

f) For which value of b would the function

$f(x) = x$ when $x < 1$

$= bx^2$ when $x \ge 1$ be continuous at x = 1?

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