(3 Hours) [Total Marks : 100]

N.B.: 1. All questions are compulsory.
2. Figures to the right indicate full marks.

Q.1 Choose correct alternative in each of the following: (20)

i. What is the GCD of 137 and 1797?
(a) 1 (b) 137
(c) 1797 (d) None of these

ii. The sum of all binomial coefficients in the expansion of $(a+b)^8$ is
(a) 100 (b) 256
(c) 1024 (d) None of these

iii. For positive integers a and b, $\gcd(a,b)\cdot lcm\,(a,b)$ is
(a) 1 (b) a
(c) ab (d) None of these

iv. Let X and Y be two non-empty sets and $f: X \to Y$ be a function. Suppose $A \subseteq X, B \subseteq Y$
(a) $f(A) \subseteq X, f^{-1}(B) \subseteq X$ (b) $f(A) \subseteq Y, f^{-1}(B) \subseteq Y$
(c) $f(A) \subseteq X, f^{-1}(B) \subseteq Y$ (d) $f(A) \subseteq Y, f^{-1}(B) \subseteq X$

v. If $A = \{a,b,c\}, B = \{1,2,3,4\}$. Which of the following is not a function?
(a) $f_1 = \{(a,1),(b,2),(c,3)\}$ (b) $f_2 = \{(a,1),(b,3),(c,3),(c,4)\}$
(c) $f_3 = \{(a,1),(b,2),(c,2)\}$ (d) None of these

vi. A relation $R = \{(1,1),(2,2),(1,2),(2,1)\}$ in set $X = \{1,2,3\}$ is
(a) R is reflexive (b) R is transitive
(c) R is symmetric (d) None of these

vii. Let $A = \{1,2,3,4\}$. Which of the following is a partition of A?
(a) {{1,2},{2,4},{3}} (b) {{1,2},{3,4}}}
(c) {{1,2},{1,4}} (d) None of these

viii. If $1-4i$is root of polynomial $f(x)$ of degree 2 then.......... is also a root
(a) $-1-4i$ (b) $-1+4i$
(c) $1+4i$ (d) None of these

ix. Degree of constant polynomial is
(a) 0 (b) 1
(c) 2 (d) None of these

x. If one of the roots of the quadratic polynomial $(k-1)x^2+kx-8$ is 2 , then the value of k is
(a) 1 (b) 2
(c) 3 (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Show that two integers a and b, not both zero have a unique positive g.c.d. which can be expressed in the form of ma + nb, where m and n are integers.

ii. Let the integer $n > 1$, has the prime factorization $n = P_1^{e_1} \dots \dots P_r^{e_r}$ then show that

$$\phi(n) = \left(P_1^{e_1} - P_1^{e_1 - 1}\right)\left(P_2^{e_2} - P_2^{e_2 - 1}\right) \dots \left(P_r^{e_r} - P_r^{e_r - 1}\right)$$

where ϕ denotes Euler's phi function.

b) Attempt any TWO questions from the following: (12)

i. For integers a, b if $(a, 4) = 2,\ (b, 4) = 2$ then prove that $(a + b, 4) = 4$.

ii. State and prove Euclid's Lemma.

iii. Find g c d of $a = 879$ and $b = 1216$ and express it in form $ma + nb$.

iv. Prove $1^3 + 2^3 + 3^3 + \cdots\cdots + n^3 = \frac{n^2(n+1)^2}{4}$ using principle of induction

Q.3 a) Attempt any ONE question from the following: (08)

i. Let f : A → B and g : B → C be bijective functions. Prove that g∘f : A→ C is also a bijective function. Is the converse true? Justify.

ii. Prove that an equivalence relation in a non-empty set gives a partition of that set.

b) Attempt any TWO questions from the following: (12)

i. Define composition of two functions. If f : ℝ → ℝ and g : ℝ → ℝ are bijective functions given by $f(x) = 4x + 3$ and $g(x) = 2x^3$ then verify that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

ii. Let f : X → Y be any function and A_1, A_2 be two non-empty subsets of X. Show that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

iii. Define a binary operation ∘ on ℤ as a∘b = ab, for a,b ∈ℤ. Check whether ∘ satisfies commutative and associative properties. Also find identity element and inverse element of a if they exist under ∘.

iv. Define following with suitable examples, a) Identity function b) Constant function c) Bijective function.

Q.4 a) Attempt any ONE question from the following: (08)

i. a) Prove that a polynomial of degree n has atmost n roots.

b) If $f(x) \in \mathbb{R}[x]$ and $\alpha \in ₵$ is a root of $f(x)$, then prove that its conjugate $\bar{\alpha}$ is also a root of $f(x)$

ii. If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial in $\mathbb{R}[x]$ with integer coefficients and a rational number $\frac{p}{q}$, p, q $\in \mathbb{Z}, (p, q) = 1, q \neq 0$, is a root of $f(x)$, prove that $p|a_0$ and $q|a_n$. Hence prove that if $f(x)$ is a monic polynomial then it's rational root is an integer.

b) Attempt any TWO questions from the following: (12)

i. Find G.C.D. of $f(x) = x^3 - 3x^2 + 2x - 6$ and $g(x) = x^3 - 2x^2 - 2x - 3$ in $\mathbb{R}[x]$.

ii. Find k in $f(x) = x^3 - 4x^2 - 4$x + k if sum of two of its roots is equal to the third root.

iii. Solve $4x^3 + 20x^2 - 23$x + 6 = 0 given that two of its roots are equal.

iv. Find the fourth roots of unity and show that their sum is zero.

Q.5 Attempt any FOUR questions from the following: (20)

a) Prove that 3|a(a+1)(a+2) for any arbitrary integer a.

b) State Wilson's theorem and verify it for prime 11.

c) If f : $\mathbb{R} - \{\frac{4}{5}\} \to \mathbb{R} - \{0\}$ such that f(x) $= \frac{1}{5x-4}$, show that f is bijective and hence find the inverse of f.

d) A relation R in $\mathbb{Z}$ is defined as " xRy iff 5x − y is divisible by 4". Show that R is a equivalence relation.

e) State Factor theorem and hence check whether g(x) = x + 1 is a factor of $f(x) = x^4 - 2x^3 + 2x^2 - 3x + 2$.

f) Find the multiplicity of the root 2 of $f(x) = x^4 - 3x^3 + 3x^2 - 8x + 12$.

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