Paper / Subject Code: 81115 / Mathematics paper I

(3 Hours) [Total Marks : 100]

N.B. **1**. All questions are compulsory.
2. Figures to the right indicate full marks
3. Use of Calculator is not allowed.

Q.1 Choose correct alternative in each of the following: (20)

i. For a, b, x in $\mathbb{R}$ if $a + x = b + x$ then this means $a = b$

a) only if $x \neq 0$ b) cannot say
c) always d) none of the above

ii. If $S = \{ x \in \mathbb{R} : |x - 7| < 1 \}$ then

a) S is bounded b) S is only bounded above
c) S is only bounded below d) cannot say

iii. Which of the following sets is a neighbourhood of -1 with radius 1, N(-1,1) in $\mathbb{R}$?

a) (0,2) b) (-2,0)
c) [0,3] d) none of these

iv. The sequence (n^2) in $\mathbb{R}$ is

a) divergent b) convergent
c) bounded d) none of these

v. The sequence (x_n) where $x_n = \frac{1}{3^n}$, $\forall n \in \mathbb{N}$ is

a) monotonic increasing b) monotonic decreasing
c) Cannot say d) none of these

vi. Every constant sequence in $\mathbb{R}$ is

a) monotonic increasing b) divergent
c) bounded d) none of these

vii. The value of $\lim_{n\to\infty} (\frac{x}{2})^n$ for $0 < x < 1$ is

a) 1 b) 0
c) -1 d) none of these

viii. The value of $\lim_{x\to\infty} \frac{x^3}{3x^3+5}$

a) 1 b) 0
c) $\frac{1}{3}$ d) none of these

ix. If $f(x) = \frac{x^2-7x+12}{x-3}$ for $x \neq 3$ then $\lim_{x\to 3} f(x)$

a) does not exist b) – 1

c) 1 d) none of these

x. The function $f(x) = x$, $x \in \mathbb{R}$ is

a) continuous everywhere b) continuous if $x > 0$

c) discontinuous everywhere d) none of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Prove that any two distinct real numbers can be separated by disjoint neighborhoods in ℝ. Hence find disjoint neighborhoods of 2.33 and 2.333.

ii. Define infimum of a non-empty set.

Prove that a lower bound m of a non-empty set S is the infimum of S iff for all $\epsilon > 0 \; \exists \; x \in S$ such that $m + \epsilon > x$.

b) Attempt any TWO questions from the following: (12)

i. Show that for any two real numbers a and b,

$$|a+b| = |a| + |b| \Leftrightarrow ab \geq 0$$

ii. Prove that for positive real numbers x and y,

(1) If $0 \leq x < y$, then $x^2 \leq xy < y^2$

(2) $x^2 + y^2 = 0 \Leftrightarrow x = 0$ and $y = 0$

iii. If A, B are bounded subsets of real numbers then prove that

$$\inf(A+B) = \inf A + \inf B$$

iv. Prove that for positive real numbers a, b and c,

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

Q.3 a) Attempt any ONE question from the following: (08)

i. Prove that a real sequence is convergent if and only if it is a Cauchy sequence.

ii. Prove that every monotone, bounded sequence of real numbers is convergent.

b) Attempt any TWO questions from the following: (12)

i. If $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and $\epsilon > 0$ is arbitrary, then prove that there exists $n_0 \in \mathbb{N}$ such that $|3a_n + 4b_n - 3a - 4b| < \epsilon \;\; \forall \; n \geq n_0$.

ii. Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n} \; \forall \, n \in \mathbb{N}$. Prove that sequence (a_n) is increasing and bounded above by 2.

iii. Let (a_n) and (b_n) be two convergent sequences such that $\lim_{n\to\infty}(3a_n + 4b_n) = 10$ and $\lim_{n\to\infty}(a_n - 2b_n) = 11$. Then find $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$.

iv. If (a_n) is a sequence of non-negative real numbers and $\lim_{n\to\infty} a_n = a$. Then prove that $a \geq 0$ and $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

Q.4 a) Attempt any ONE question from the following: (08)

i. *Let* $f, g : \mathbb{R} \to \mathbb{R}$ and $l, m \in \mathbb{R}$. If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ then prove that $\lim_{x\to a}[f(x) - g(x)] = l - m$.

ii. Let f : ℝ → ℝ be a function and p∈ ℝ. If ($f(x_n)$) converges to f(p) for any sequence (x_n) converging to p then prove that f is continuous at p.

b) Attempt any TWO questions from the following: (12)

i. Draw the graph of the function f where f(x) = | x | + 3 for -3 ≤ x ≤ 3.

ii. Show that $\lim_{x\to 3}(14 - 2x) = 8$ using $\epsilon - \delta$ definition.

iii. State and prove Sandwich theorem for limit of functions in ℝ.

iv. Let f : ℝ → ℝ be a function which is continuous at p ∈ ℝ. Then prove that there exists $\delta > 0$ and M > 0 such that | f(x) | ≤ M , for all x ∈ $N(p, \delta)$.

Q.5 Attempt any FOUR questions from the following: (20)

a) Show that $x + \frac{1}{x} \geq 2$ for $x > 0$.

b) Let A and B be nonempty bounded subsets of ℝ such that $A \subseteq B$. Prove that $sup\ A \leq sup\ B$.

c) Let $x_n = \cos\left(\frac{n\pi}{2}\right), \forall\, n \in \mathbb{N}$. Show that (x_n) is not convergent by exhibiting two convergent subsequences of (x_n) converging to two different limits.

d) Show that $\lim_{n\to\infty} \frac{\sin n}{n^2} = 0$ using Sandwich theorem.

e) Let $f, g: \mathbb{R} \to \mathbb{R}$ be functions and $p \in \mathbb{R}$. If $\lim_{x\to p} f(x) = l$ and $\lim_{x\to p} g(x) = m$ and $f(x) \geq g(x), \forall\, x \in \mathbb{R}$ then show that $l \geq m$.

f) Find the value of b so that f becomes continuous at $\frac{\pi}{2}$ where $(x) = \begin{cases} -\sin x, & x < \frac{\pi}{2} \\ bx^2, & x \geq \frac{\pi}{2} \end{cases}$.

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