Q.P.Code: 30030

(3 Hours) [Total Marks : 100

N.B.: 1. All questions are compulsory.
2. Figures to the right indicate full marks.

Q.1 Choose correct alternative in each of the following: (20)

i. For $x, y, z \in \mathbb{R}$ if $xy = xz$ then
(a) $y = z$
(b) $y = z$ only if $x \neq 0$
(c) $x = 0$
(d) None of these.

ii. For $x, y \in \mathbb{R}$ if $xy = 0$ then
(a) $x = 0$ and $y = 0$
(b) $x = 0$ or $y = 0$
(c) $x - y = 0$
(d) $x = y$

iii. Between any two distinct real numbers there always exists
(a) Only rational numbers
(b) Both rational and irrational numbers
(c) Only irrational numbers
(d) None of these.

iv. Every nonempty subset of $\mathbb{R}$ which is bounded below has
(a) glb in $\mathbb{R}$
(b) lub in $\mathbb{R}$
(c) glb and lub in $\mathbb{R}$
(d) Neither glb nor lub in $\mathbb{R}$

v. The sequence (x_n) where $x_n = \frac{1}{3n}$,$\forall\, n \in \mathbb{N}$ is
(a) convergent
(b) divergent
(c) unbounded
(d) None of these.

vi. The sequence (x_n) where $x_n = n^3$,$\forall\, n \in \mathbb{N}$ is
(a) convergent
(b) monotonic increasing
(c) Cauchy
(d) bounded

vii. The value of $\lim_{n \to \infty} 3x^n$ for $0 < x < \frac{1}{4}$ is
(a) 1
(b) 2
(c) -1
(d) 0

viii. The value of $\lim_{x \to 0^+} \frac{|x|}{x}$
(a) 1
(b) Does not exist
(c) -1
(d) None of these

Turn Over

Q.P.Code: 30030

ix. Which of the following function has removable discontinuity at $x = 3$?

(a) $\frac{x^2-7x+12}{x-3}$ (b) $\cos\left(\frac{1}{x-3}\right)$

(c) $\frac{x^2+10x+25}{x-3}$ (d) None of these

x. The function $f(x) = \sin(2x+3)$, $x \in \mathbb{R}$ is

(a) Continuous everywhere (b) Discontinuous at $x = -\frac{3}{2}$

(c) Continuous only for $x \geq 0$ (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Prove that a nonempty subset of $\mathbb{R}$ which is bounded below has glb in $\mathbb{R}$.

ii. If x and y are any real numbers with $x < y$ then prove that there exists a rational number r such that $x < r < y$.

b) Attempt any TWO questions from the following: (12)

i. Let S be a nonempty subset of $\mathbb{R}$ which is bounded above. Prove that lub S is unique.

ii. Let S be a nonempty subset of $\mathbb{R}$. For $a \in \mathbb{R}$, define $aS = \{as : s \in S\}$. If S is bounded above then prove that lub $aS = a$ lub S if $a > 0$.

iii. If $x > 0$ then show that there exists $m \in \mathbb{N}$ such that $m - 1 \leq x < m$.

iv. State and prove Hausdorff property of $\mathbb{R}$.

Q.3 a) Attempt any ONE question from the following: (08)

i. Define subsequence of a sequence in $\mathbb{R}$. Prove that every subsequence of a convergent sequence is convergent.

ii. Prove that every monotonic decreasing sequence of real numbers is convergent if it is bounded below.

b) Attempt any TWO questions from the following: (12)

i. Prove that the limit of sequence (x_n) is $\frac{2}{3}$ using definition where $x_n = \frac{2n+3}{3n+2}$, $\forall\, n \in \mathbb{N}$

Turn Over

Q.P.Code: 30030

ii. Prove that every Cauchy sequence of real numbers is bounded.

iii. Let (x_n) and (y_n) be two convergent sequences of real numbers converging to p and q respectively. Prove that the sequence $(x_n - y_n)$ converges to $p - q$.

iv. Show that the sequence (x_n) is divergent where $x_n = (-1)^n, \forall\, n \in \mathbb{N}$.

Q.4 a) Attempt any ONE question from the following: (08)

i. Let $f, g: \mathbb{R} \to \mathbb{R}$ be two functions and let $a \in \mathbb{R}$, if $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then prove that $\lim_{x \to a} (9f - 2g)(x) = 9l - 2m$.

ii. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function at $p \in \mathbb{R}$. Prove that sequence $(f(x_n))$ converges to $f(p)$ for any sequence (x_n) converging to p.

b) Attempt any TWO questions from the following: (12)

i. Prove that $\lim_{x \to 6} (4x + 6) = 30$ using $\epsilon - \delta$ definition of limit.

ii. Draw graph of a function e^{x+2} for $x \in \mathbb{R}$.

iii. Let $f: \mathbb{R} \to \mathbb{R}$ be a function and let $l \in \mathbb{R}$. Give definition of $\lim_{x \to \infty} f(x) = l$ and also find $\lim_{x \to \infty} \frac{x^4 + 5}{5x^4}$.

iv. Discuss the continuity of the following function at $x = 6$ where $f(x) = \begin{cases} x^2 + 10 & if \;\; x < 6 \\ x - 20 & if \; 6 \le x \end{cases}$

Q.5 Attempt any FOUR questions from the following: (20)

a) State and prove the Arithmetic-Geometric mean inequality for $a, b \in \mathbb{R}$.

b) Prove that $\big||x| - |y|\big| \le |x - y|$ for all $x, y \in \mathbb{R}$.

Turn Over

c) Let $x_n = \frac{1}{5n}$, $\forall\, n \in \mathbb{N}$. Show that (x_n) is a Cauchy sequence.

d) Show that (x_n) converges to 3 using definition where $x_n = 3, \forall\, n \in \mathbb{N}$.

e) Let $f: \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Prove that if $\lim_{x \to a} f(x) = l$ then $\lim_{x \to a} |f(x)| = |l|$. Is the converse true? Justify your answer.

f) State Sandwich theorem for limit of function. Use it to find $\lim_{x \to 4} f(x)$ where $1 - \frac{(x-4)^2}{4} \le f(x) \le 1 + \frac{(x-4)^2}{4}$ for all $x \in \mathbb{R}$.

****