Algebra - I
20/10/2015

QP Code : 12935

2 $\frac{1}{2}$ Hours]

[Total Marks: 75

N.B.: (1) All questions are compulsory.
(2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any ONE

i. State and prove the Cayley Hamilton theorem. (8)

ii. Let V be a finite dimensional inner product space over $\mathbb{R}$ and $T: V \to V$ be a linear transformation. Prove that the following statements are equivalent. (8)

(p) T is orthogonal.

(q) $\|T(x)\| = \|x\|$ for all $x \in V$.

(b) Answer any TWO

i. For real vector space V and a subspace W of V define quotient space V/W. Let V be a finite dimensional real vector space and W be a subspace of V. Show that $dimV/W = dimV - dimW$. (6)

ii. Let V be a finite dimensional inner product space over $\mathbb{R}$. If $f: V \to V$ is an isometry, then show that there exists unique $x_0 \in V$ and an orthogonal linear transformation $T: V \to V$ such that $f(x) = T(x) + x_0, \quad \forall x \in V$. (6)

iii. Find an orthogonal transformation in $\mathbb{R}^3$ which represents reflection with respect to $x - y + z = 0$. (6)

iv. Show that a 2×2 orthogonal matrix A with $det A = 1$ is a matrix of rotation. (6)

2. (a) Answer any ONE

i. Define algebraic multiplicity and geometric multiplicity of an eigen value λ of a real matrix A. Show that, if A is diagonalizable then (a) algebraic and geometric multiplicity of each eigen value of A coincide (b) sum of geometric multiplicity of all the eigen values of A is n. (8)

ii. Show that every real symmetric matrix is orthogonally diagonalizable. (8)

(b) Answer any TWO

i. Show that every quadratic form $Q[x] = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ over $\mathbb{R}$ can be reduced to standard form $\sum_{i=1}^{n} \lambda_i y_i^2$ by orthogonal change of variable $X = PY$, where X, Y are column vectors of $\mathbb{R}^n$. (6)

ii. Let A be $n \times n$ upper triangular real matrix. (6)

(p) If all the main diagonal entries of A are distinct, then show that A is diagonalizable.

(q) If each main diagonal entries of A is λ and A is diagonalizable then show that $A = \lambda I_n$.

[P.T.O.]

CM-Con. 2782-15.

iii. Find a square matrix A of order 3 which has eigen values 0,1,−1 with corresponding eigen vectors $(0,1,-1)^t$, $(1,-1,1)^t$ and $(0,1,1)^t$. (6)

iv. Show that an $n \times n$ real symmetric matrix is positive definite if and only if all its eigen values are positive. (6)

3. (a) Answer any ONE (8)

i. G be a finite cyclic group of order n. Prove that G contains a unique subgroup of order d, for every divisor d of n.

ii. Let G be a group and H and K be any two subgroup of G. Show that HK be subgroup of G if and only if $HK = KH$.

(b) Answer any TWO

i. If $f : G \to G'$ is a group homomorphism then define kernel of f and prove that kernel of f is a subgroup of G.

ii. Let G, G' be groups and $f : G \to G'$ be a onto homomorphism of groups. Show that

(p) If G is abelian then G' is abelian.

(q) G is cyclic and $G = \langle a \rangle$ then G' is cyclic and $G' = \langle f(a) \rangle$.

iii. Let $G = \{\overline{5}, \overline{15}, \overline{25}, \overline{35}\}$ under multiplication of residue classes mod 40. Form composition table of G. State identity element of G and show that G is group.

iv. Give an example of a group G such that $o(a) = 2$, $o(b) = 2$ and $o(ab) = 5$.

4. Answer any THREE

(a) Let A, B be $n \times n$ real matrices. If A and AB are orthogonal then prove that B and BA are both orthogonal matrices.

(b) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 2 & 4 \end{bmatrix}$. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(X) = AX$ (X being a column vector sin $\mathbb{R}^3$). Find $kerT$ and ImT. Verify the fundamental theorem of homomorphism of vector spaces incase of T.

(c) Determine whether matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 24 \end{pmatrix}$ is diagonalizable or not.

(d) Show that eigen vector associated with distinct eigen values of real symmetric matrix are orthogonal.

(e) Show that a group G is abelian if and only if $f : G \to G$ define as $f(x) = x^2$ is a group homomorphism.

(f) If G is a finite group and H is a nonempty subset of G then prove that H is subgroup of G if and only if for any $a, b \in H$, $ab \in H$.

****