

(2½ Hours)

[Total Marks: 75]

- N. B.: (1) **All** questions are **compulsory**.
 (2) Make **suitable assumptions** wherever necessary and **state the assumptions** made.
 (3) Answers to the **same question** must be **written together**.
 (4) Numbers to the **right** indicate **marks**.
 (5) Draw **neat labeled diagrams** wherever **necessary**.
 (6) Use of **Non-programmable** calculators is **allowed**.

1. Attempt any three of the following:

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- Define Universal Existential Statement and Existential Universal Statement. Give examples of each.
- Define Cartesian product. Let $\mathbf{R}$ denote the set of all real numbers. Describe $\mathbf{R} \times \mathbf{R}$.
- Find the number of integers between 1 and 250 that are divisible by 2 or 3 or 5 or 7.
- Prove that $(A \cup B) \cap (A \cap B)' = (A - B) \cup (B - A)$
- Write the negation of each of the following statements as simply as possible:
 - If she works, she will earn money.
 - He swims if and only if the water is warm.
 - If it snows, then they do not drive the car.
 - John is 6 feet tall and he weighs at least 120 Kg.
 - The train was late or Amol's watch was slow.
- Define the following:
 - Argument, Premises
 - Syllogism
 - Explain Modus Ponens and Modus Tollens with examples.

2. Attempt any three of the following:

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- Let

$Q(n)$ be " n is a factor of 8,"
 $R(n)$ be " n is a factor of 4,"
 $S(n)$ be " $n < 5$ and $n \neq 3$,"

and suppose the domain of n is $\mathbf{Z}^+$, the set of positive integers. Use the $\Rightarrow$ and $\Leftrightarrow$ symbols to indicate true relationships among $Q(n)$, $R(n)$, and $S(n)$.
- Define *necessary and sufficient conditions* and *only if* as applied to universal conditional statements. Rewrite the following statements as formal and informal quantified conditional statements. Do not use the word necessary or sufficient.
 - Squareness is a sufficient condition for rectangularity.
 - Being at least 35 years old is a necessary condition for being President of the United States.
- A college cafeteria line has four stations: salads, main courses, desserts, and beverages. The salad station offers a choice of green salad or fruit salad; the main course station offers spaghetti or fish; the dessert station offers pie or cake; and the beverage station offers milk, soda, or coffee. Three students, Uta, Tim, and Yuen, go through the line and make the following choices:
 Uta: green salad, spaghetti, pie, milk
 Tim: fruit salad, fish, pie, cake, milk, coffee
 Yuen: spaghetti, fish, pie, soda

Write each of following statements informally and find its truth value.

- i. $\exists$ an item I such that $\forall$ students S , S chose I .
- ii. $\exists$ a student S such that $\forall$ items I , S chose I .
- iii. $\exists$ a student S such that $\forall$ stations Z , $\exists$ an item I in Z such that S chose I .
- iv. $\forall$ students S and $\forall$ stations Z , $\exists$ an item I in Z such that S chose I .
- d. Define a prime number and composite number. Give symbolic definitions of the same. Disprove the following by giving two counter examples:
 - i. For all real numbers a and b , if $a < b$ then $a^2 < b^2$.
 - ii. For all integers n , if n is odd then $(n-1)/2$ is odd.
 - iii. For all integers m and n , if $2m+n$ is odd then m and n are both odd.
- e. Define divisibility. Hence prove that for all integers a , b , and c , if $a \mid b$ and $a \mid c$ then $a \mid (b+c)$ and $a \mid (b-c)$.
- f. Use the quotient-remainder theorem with $d = 3$ to prove that the product of any three consecutive integers is divisible by 3. Use the mod notation to rewrite the result

3. Attempt any three of the following:

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- a.
 - i. Write the following as a single summation:

$$3 \sum_{k=1}^n (2k-3) + \sum_{k=1}^n (4-5k)$$
 - ii. Write the following as a single product:

$$\left(\prod_{k=1}^n \frac{k}{k+1} \right) \cdot \left(\prod_{k=1}^n \frac{k+1}{k+2} \right)$$
 - iii. Find $1(1!!) + 2(2!!) + 3(3!!) + \dots + m(m!!)$; $m = 2$
 - iv. Find

$$\left(\frac{1}{1+1} \right) \left(\frac{2}{2+1} \right) \left(\frac{3}{3+1} \right) \dots \left(\frac{k}{k+1} \right); k = 3$$
 - v. Prove that for all nonnegative integers n and r with $r+1 \leq n$,

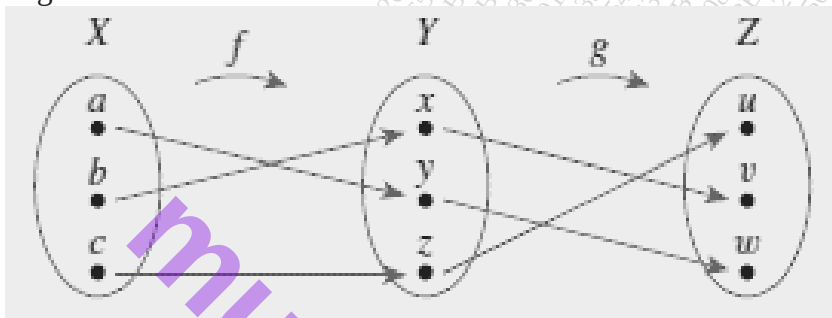
$$\binom{n}{r+1} = \frac{n-r}{r+1} \binom{n}{r}$$
- b. Prove that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 $\forall n \in \mathbb{N}$
- c. Determine the sequence whose recurrence relation is $a_n = 4a_{n-1} + 5a_{n-2}$ with $a_1 = 2$ and $a_2 = 6$
- d.
 - i. Define $G: J_5 \times J_5 \rightarrow J_5 \times J_5$ as follows: For all $(a, b) \in J_5 \times J_5$,

$$G(a, b) = ((2a+1) \bmod 5, (3b-2) \bmod 5)$$
 Find: $G(4, 4)$, $G(2, 1)$, $G(3, 2)$, $G(1, 5)$
 - ii. Let F and G be functions from the set of all real numbers to itself. Define the product functions $F \cdot G: \mathbb{R} \rightarrow \mathbb{R}$ and $G \cdot F: \mathbb{R} \rightarrow \mathbb{R}$ as follows: For all $x \in \mathbb{R}$,

$$(F \cdot G)(x) = F(x) \cdot G(x)$$

$$(G \cdot F)(x) = G(x) \cdot F(x)$$
 Does $F \cdot G = G \cdot F$? Explain.

- e.
- Define Floor: $\mathbf{R} \rightarrow \mathbf{Z}$ by the formula $Floor(x) = \lfloor x \rfloor$, for all real numbers x .
 - Is Floor one-to-one? Prove or give a counterexample.
 - Is Floor onto? Prove or give a counterexample.
 - Let S be the set of all strings of 0's and 1's, and define $l: S \rightarrow \mathbf{Z}^{nonneg}$ by $l(s) =$ the length of s , for all strings s in S .
 - Is l one-to-one? Prove or give a counterexample.
 - Is l onto? Prove or give a counterexample.
- f. Let $X = \{a, c, b\}$, $Y = \{x, y, z\}$, and $Z = \{u, v, w\}$. Define $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ by the arrow diagrams below.

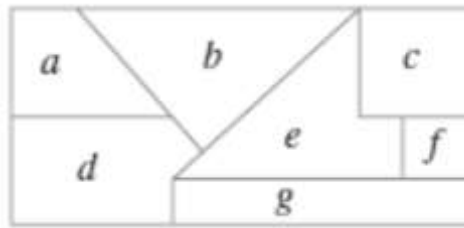


Find: $g \circ f, (g \circ f)^{-1}, f^{-1}, g^{-1}, f^{-1} \circ g^{-1}$.
How $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are related?

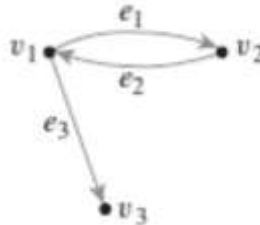
4. **Attempt any three of the following:**

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- a Draw the directed graph for the following relations:
- A relation R on $A = \{0, 1, 2, 3\}$ by $R = \{(0, 0), (1, 2), (2, 2)\}$.
 - Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows:
 $\text{For all } x, y \in A, x R y \Leftrightarrow x \mid y$.
- b Determine whether the following relations are reflexive, symmetric, transitive or none of these. Justify your answer.
- R is the "greater than or equal to" relation on the set of real numbers:
 $\text{For all } x, y \in \mathbf{R}, x R y \Leftrightarrow x \geq y$.
 - D is the relation defined on $\mathbf{R}$ as follows:
 $\text{For all } x, y \in \mathbf{R}, x D y \Leftrightarrow xy \geq 0$.
- c Let $\mathbf{R}$ be the set of all real numbers and define a relation R on $\mathbf{R} \times \mathbf{R}$ as follows: For all (a, b) and (c, d) in $\mathbf{R} \times \mathbf{R}$, $(a, b) R (c, d) \Leftrightarrow$ either $a < c$ or both $a = c$ and $b \leq d$.
Is R a partial order relation? Prove or give a counterexample.
- d Imagine that the diagram shown below is a map with countries labeled a – g . Is it possible to color the map with only three colors so that no two adjacent countries have the same color? To answer this question, draw and analyze a graph in which each country is represented by a vertex and two vertices are connected by an edge if, and only if, the countries share a common border.



- e i. Find the adjacency matrix of the following graph:



- ii. Find directed graphs that have the following adjacency matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

- f For the following either draw the graph as per the specifications or explain why no such graph exists:

- Graph, circuit-free, nine vertices, six edges
- Tree, six vertices, total degree 14
- Tree, five vertices, total degree 8
- Graph, connected, six vertices, five edges, has a nontrivial circuit
- Graph, two vertices, one edge, not a tree

5. Attempt any three of the following:

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- There are four bus lines between A and B and three bus lines between B and C. In how many ways can a man travel
 - by bus from A to C by way of B?
 - round-trip by bus from A to C by way of B?
 - round-trip by bus from A to C by way of B if he does not want to use a bus line more than once?
- How many ways can the letters of the word ALGORITHM be arranged in a row?
 - How many ways can the letters of the word ALGORITHM be arranged in a row if A and L must remain together (in order) as a unit?
 - How many ways can three of the letters of the word ALGORITHM be selected and written in a row?
 - How many ways can six of the letters of the word ALGORITHM be selected and written in a row if the first letter must be A?
 - How many ways can the letters of the word ALGORITHM be arranged in a row if the letters GOR must remain together (in order) as a unit?

- c.
 - i. If 4 cards are selected from a standard 52-card deck, must at least 2 be of the same suit? Why?
 - ii. If 5 cards are selected from a standard 52-card deck, must at least 2 be of the same suit? Why?
 - iii. A small town has only 500 residents. Must there be 2 residents who have the same birthday? Why?
 - iv. Given any set of four integers, must there be two that have the same remainder when divided by 3? Why?
 - v. Given any set of three integers, must there be two that have the same remainder when divided by 3? Why?
- d.
 - i. How many distinguishable ways can the letters of the word *HULLABALOO* be arranged in order?
 - ii. How many distinguishable orderings of the letters of *HULLABALOO* begin with U and end with L?
 - iii. How many distinguishable orderings of the letters of *HULLABALOO* contain the two letters HU next to each other in order?
- e. A bakery produces six different kinds of pastry, one of which is eclairs. Assume there are at least 20 pastries of each kind.
 - i. How many different selections of twenty pastries are there?
 - ii. How many different selections of twenty pastries are there if at least three must be eclairs?
 - iii. How many different selections of twenty pastries contain at most two eclairs?
- f. A drug-screening test is used in a large population of people of whom 4% actually use drugs. Suppose that the false positive rate is 3% and the false negative rate is 2%. Thus a person who uses drugs tests positive for them 98% of the time, and a person who does not use drugs tests negative for them 97% of the time.
 - i. What is the probability that a randomly chosen person who tests positive for drugs actually uses drugs?
 - ii. What is the probability that a randomly chosen person who tests negative for drugs does not use drugs?